The Determination of Asteroid Proper Elements

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Following a brief historical introduction, we first demonstrate that proper elements are quasi-integrals of motion and show how they are used to classify asteroids into families and to study the long-term dynamics of asteroids. Then, we give a complete overview of the analytical, semianalytical and synthetic theories for the determination of proper elements of asteroids, with a special emphasis on the comparative advantages/disadvantages of the methods, and on the accuracy and availability of the computed proper elements. We also discuss special techniques applied in some particular cases (mean motion and secular resonant bodies). Finally, we draw our conclusions and suggest directions for future work.

1. INTRODUCTION

The computation of asteroid proper elements is certainly one of the fields of asteroid research which underwent the most remarkable development in the last decade. The accuracy and efficiency of the methods introduced in this period improved dramatically. Thus we were able to solve many problems which puzzled researchers in the previous times. We could also recognize and investigate an entire spectrum of new problems, from novel classes of dynamical behavior to different phenomena which were previously either completely unknown or impossible to investigate with the available tools.

The history, definition, and applications of proper elements are described in great detail in a number of reviews (e.g. *Valsecchi et al.*, 1989; *Shoemaker at al.* 1989; *Lemaitre*, 1993; *Knežević and Milani*, 1994; *Knežević*, 1994). However, for the sake of completeness, these topics are tackled in the following sections.

1.1 Historical overview

A classical definition states that proper elements are quasi-integrals of motion, and that they are, therefore, nearly constant in time. Alternatively, one can say that they are true integrals, but of a conveniently simplified dynamical system. In any case, the proper elements are obtained as a result of the elimination of short and long periodic perturbations from their instantaneous, osculating counterparts, and thus represent a kind of "average" characteristics of motion.

A concept of proper elements has been introduced by *Hirayama* (1918) in his celebrated paper in which he announced the discovery of asteroid families. Even if not using the technical term "proper", he employed Lagrange's classical linear theory of asteroid secular perturbations to demonstrate that certain asteroids tend to cluster around special values of the orbital elements, which very closely correspond to the constants of integration of the solutions of the equations of their motion, that is, to a sort of averaged characteristics of their motion over very long time spans. In his later papers *Hirayama* (1923, 1928) explicitly computed just the proper elements (proper semimajor axis, proper eccentricity and proper inclination), and used them for the classification of asteroids into families.

The next important contribution is due to *Brouwer* (1951). He computed asteroid proper elements again using a linear theory of secular perturbations, but in combination with an improved theory of motion of the perturbing planets (*Brouwer and Van Woerkom*, 1950). By including more accurate values of planetary masses, and the effect of the "great inequality" of Jupiter and Saturn, he was able to get a more realistic value for the precession rate of the perihelion of Saturn.

Williams (1969) developed a semianalytic theory of asteroid secular perturbations which does not make use of a truncated development of the perturbing function, and which is, therefore, applicable to asteroids of with arbitrary eccentricity and inclination. Williams' proper eccentricity and proper inclination are defined as values acquired when the argument of perihelion $\omega = 0$ (thus corresponding to the minimum of eccentricity and the maximum of inclination over a cycle of ω). The theory is linearized in the planetary masses, eccentricities and inclinations, so that the proper elements computed by means of this theory (*Williams*, 1979; 1989), even if much better than the previously available ones, were still of limited accuracy.

Another achievement to be mentioned is that by *Kozai* (1979), who used his theory of secular perturbations for high inclination asteroids (*Kozai*, 1962) to define a set of proper parameters to identify the families. The selected parameters were semimajor axis, $z-$component of the angular momentum (integral of motion in a first order theory, with perturbing planets moving on circular, planar orbits), and the minimum value of inclination over the cycle of the argument of perihelion (corresponding to $\omega = \pi/2$).

Finally, we quote the work by *Schubart* (1982; 1991), *Bien and Schubart* (1987) and *Schubart and Bien* (1987), pioneering the attempts to determine proper parameters for resonant groups, that is for Hildas and Trojans. Since the usual averaging methods do not apply in this case, they adopted slightly different definitions of the proper parameters, the most important difference being the substitution of a representative value measuring the libration of the critical argument instead of the usual proper semimajor axis.

1.2 What are the proper elements/parameters ?

The notion of proper elements is based upon the linear theory of secular perturbations, which goes back to Lagrange. Linear theory neglects the short periodic perturbations, containing the anomalies in the arguments; this results in a constant semimajor axis which becomes the first proper element a_p. The long term evolution of the other variables is obtained by approximating the 'secular' equations of motion with a system of linear differential equations. Because of the linearity assumption, the solutions can be represented in the planes $(k, h) = (e \cos \varpi, e \sin \varpi)$ as the sum of 'proper modes', one for each planet, plus one for the asteroid. Thus the solution can be represented by epicyclic motion: for the asteroid, the sum of the contributions from the planets represents the 'forced' term, while the additional circular motion is the so called 'free oscillation' and its amplitude is the proper eccentricity e_p. The same applies to the $(q, p) = (\sin I \cos \Omega, \sin I \sin \Omega)$ plane, with amplitude of the free term given by the (sine of) proper inclination $\sin I_p$. Figure 1 shows the output of a numerical integration of an asteroid's orbit for $20,000$ years plotted in the (k,h) plane, and an epicyclic model fitting to the data. As it is apparent from the figure, the approximation of linear secular perturbation theory is good enough for a time span of the order of the period of circulation for the longitude of perihelion ϖ. However, even over such a time span the linear theory is only an approximation, and over a much longer time span (e.g., millions of years) it would be a rather poor approximation in most cases.

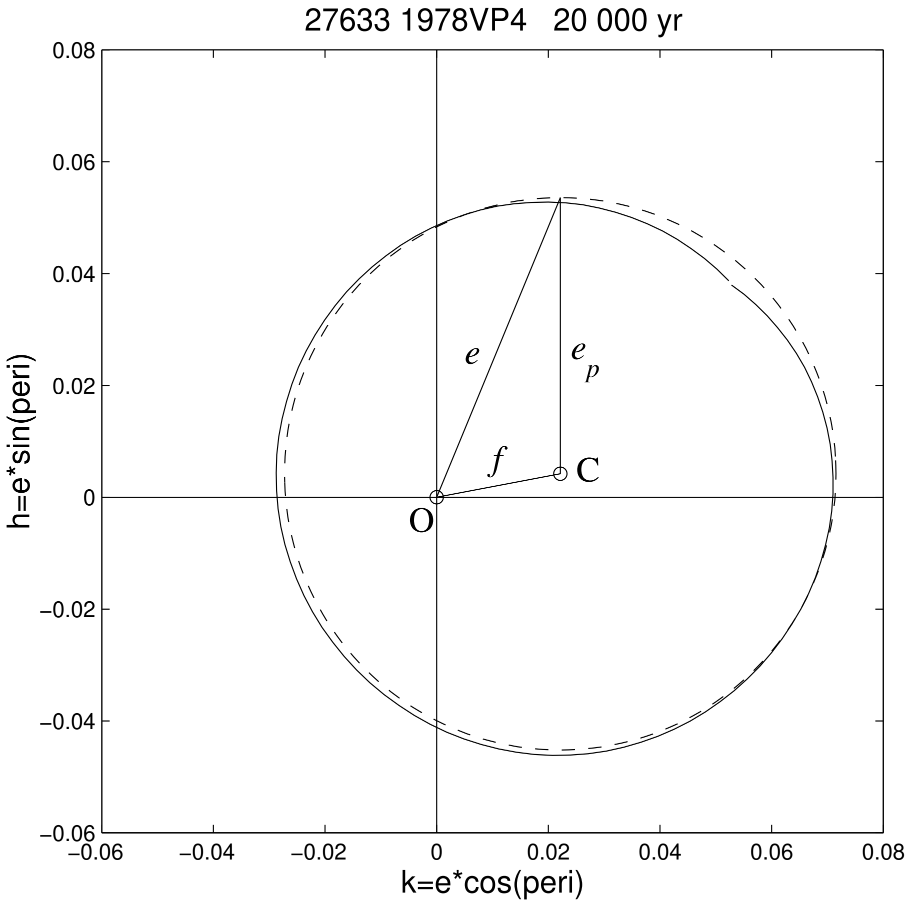

Fig. 1: The orbit of the numbered asteroid (27633) over $20,000$ years projected in the (k,h) plane (full line). The data have been digitally smoothed to remove short periodic perturbations. The point C represents the average value of the forced term, f is the forced eccentricity, and the dotted circle of radius e_p represents the best fitting epicycle. The value of the eccentricity e, obtained as length of the vectorial sum of the forced and the free terms, is an approximation of the current value.

Proper elements can also be otained from the output of a numerical integration of the full equations of motion: the simplest method is to take averages of the action-like variables a, e, I, over times much longer than the periods of circulation of the corresponding angular variables. However, this method provides proper elements of low reliability: the dynamical state can change for unstable orbits and in such a case the simple average wipes out this essential information. Thus, if the goal is to compute proper elements stable to 1% of their value or better, over time spans of millions of years, it is necessary to use much more complicated theories. This is equivalent to say that the approximative dynamical system, rather than a simple linear one, is by itself complex, even if it admits integrals which are used as proper elements.

Whatever the type of theory, on the other hand, if it is to be accurate enough to represnt the dynamics in the framework of a realistic model, its full-detail description requires to dwell into a very long list of often cumbersome technicalities. For this reason in the present paper we only give a qualitative description of the computational procedures, and then proceed to discuss the quality of the results.

Several different sets of proper parameters have been introduced over the time, but the most common set, usually referred to as "proper elements", includes proper semimajor axis a_p, proper eccentricity e_p, (sine of) proper inclination $(\sin)i_p$, proper longitude of perihelion ϖ_p and proper longitude of node Ω_p, the latter two angles being accompanied with their precession rates - the fundamental frequencies g and s, respectively.

The analytical theories and the already mentioned theory by *Williams* (1969) use a different definition of proper eccentricity and inclination. Other authors introduced even completely different parameters to replace the standard proper elements. Still, the common feature of all these parameters is their supposed near constancy in time (or more precisely, stability over very long time spans), and one can say that in this sense the term "proper" is practically a synonym for "invariable".

As explained by *Lemaitre* (1993), even if quite different in terms of the proper parameters and the ways to compute them, all the theories follow several basic steps, which can be summarized as follows:

(i) modelling of the asteroid motion (N-body, restricted 3-body), and distinguishing the fast and slow angles (that is, separating the perturbations depending on the mean longitudes from the rest);

(ii) removal of the short periodic perturbations (analytical or numerical averaging, on-line filtering) and computation of the mean elements;

(iii) splitting of the resulting Hamiltonian into two parts – the integrable (secular) part, and the perturbation (the long-periodic part depending on slow angles). In the case of the synthetic theories this step corresponds to the removal of the forced terms;

(iv) removal of the long-periodic terms and computation of the proper values. At this stage, analytic and semianalytic theories resort to the averaging over the slow angles and to the iterative procedures to compute the inverse map of canonical transformation, while in the case of synthetic theories this phase includes the Fourier analysis and extraction of the principal harmonics from the time series of mean elements (freed from the forced terms).

(v) output of the proper values, possibly accompanied with their errors (standard deviations, maximum excursions), fundamental frequencies, quality and resonant codes, chaotic behavior indicators and other information.

1.3 Applications.

For a long time proper elements have been used for the sole purpose of identifying the asteroid families. Even today this is one of their most common applications (*Zappalà et al.* 1990, 1995). Since family identification is the subject of another paper in this book (*Bendjoya et al.*, 2002), we are not going to discuss it here in further detail.

On the other hand, the proper elements are computed for very different asteroids - from NEA's (*Gronchi and Milani*, 2001) to Trojans (*Milani*, 1994; *Beaugé and Roig*, 2001). They are used to describe very different dynamical phenomena, like stable chaos (*Milani et al.*, 1997), resonant dynamics (*Morbidelli*, 1993), or nongravitational effects (*Farinella and Vokrouhlický*, 1999). The list of problem-solving applications of proper elements is long, ranging from the determination of locations of secular resonances in the asteroid belt (*Milani and Knežević*, 1994) and in a wider planetary region (*Knežević et al.* 1991, *Michel and Froeschlé*, 1997), through physical and dynamical studies of asteroid families

(*Morbidelli et al.* 1995) and the determination of their age (*Milani and Farinella*, 1994; *Knežević* 1999), to such "exotic" uses as the study of an optimum strategy for the orbit maintenance of a low lunar polar orbiter (*Knežević and Milani*, 1998), to mention but a few of them.

2. THE THEORIES

2.1 Analytical theory

The main problem in the manufacturing of fully analytical theories of proper elements is in the complete *degeneracy* of the unperturbed dynamics, the 2-body problem. Degeneracy means that some of the fundamental frequencies are zero, and indeed in the 2-body approximation both the perihelia and the nodes do not precess at all. In the Hamiltonian formalism, this is expressed by the statement that the unperturbed Hamiltonian function $H_0 = H_0(L)$ is a function only of one variable L, in turn a function of the semimajor axis. The perturbed problem with Hamiltonian $H_0(L) + \mu H_1(\ell, g, L, G)$ (the small parameter μ representing the ratio of the mass of the planets to the mass of the Sun) can be handled with different perturbative approaches, but they all have in common the use of a solution of the unperturbed 2-body problem to be substituted into the perturbing function H_1. Thus they also have in common the problem that the angle variable ℓ, the mean anomaly conjugate to L, can be eliminated, but the angles g, conjugate to the other action variables not appearing in H_0, cannot be removed by averaging.

In practice, this implies that the procedure to compute proper elements must always be decomposed into two computational steps: the transformation from osculating orbital elements to mean elements, free from the short periodic perturbations (with arguments containing the fast variable ℓ), and the transformation of the mean elements into proper elements. A fully analytical theory performs both steps by means of the computation of functions for which analytical expressions, in practice truncations of some series, are available. Note that it is possible, sometimes even convenient, to mix two different methods: as an example, *Lemaitre and Morbidelli* (1994) use mean elements computed analytically as a starting point for their semianalytic computation of proper elements.

Different perturbation techniques can be used, the *Lie series* technique being the most conveninent for theories pushed to higher order (and therefore based upon series with many terms). In essence, analytical perturbation techniques exploit the approximation of the perturbing function H_1 by means of a finite sum of terms, each with a simple expression of the form

$$\mu\, b(L)\, e^h e'^k I^j I'^m \cos(p\ell + q\ell' + \delta)$$

where h, k, j, m, p, q are integers, the primed elements refer to some perturbing planet, b is a known function and δ is some combination (with integer coefficients) of the angles included in g, the perihelia and nodes of the asteroid and the planet. The truncation of the series is mostly based upon the degree in the small parameters eccentricity and inclination (of both the asteroid and the planets), although truncation for large values of the integers p, q is also possible. Thus we can describe the degree of completeness of a theory by means of the order (in the small parameter μ) and of the degree (in the eccentricities and inclinations). From this arises the main limitation of the analytical method: the accuracy, and stability with time, of the proper elements decreases as the asteroid eccentricity and inclination increase. There is a boundary between the region where the analytical proper elements are most suitable and the region where more computationally intensive methods, such as the semianalytic ones, need to be used; *Knežević et al.* (1995) have mapped this boundary (see Section 2.4). The simple analytical form of the terms implies that it is possible to perform both derivatives and integrals analytically; thus the corresponding operations can be applied to the series term by term. An analytical theory can be expressed by means of derivatives, integrals and arithmetic operations on these series; thus they can be explicitly computed by means of a finite, although large, number of elementary operations. In practice, the current analytical theories use series with several tens of thousands of terms.

The series used in the current theories are essentially based upon the expansions computed by *Yuasa* (1973) and corrected and completed by *Knežević* (1989, 1993). These are complete to degree 4 in eccentricities and inclinations; only a few special terms of degree 6 have been added later (*Milani and Knežević* 1994). *Yuasa* (1973) defined an algorithm to compute proper elements with a theory containing the main terms of order 2 in μ and complete to degree 4 in eccentricity and inclination, but one essential step was missing. *Milani and Knežević* (1990) found that, at orders > 1 in μ, the formulas of perturbation theory explicitly provide a map between proper and mean elements, in the opposite direction from the one which we deal with in practice. Thus the computation of proper elements from mean elements requires the solution of an inverse function problem, and this is possible only by an iterative procedure. Later *Milani and Knežević* (1999) applied the same argument to the computation of mean elements from osculating ones.

There are two reasons to use analytical proper elements, even when other methods (for example, the synthetic ones) are available for the same objects.

First, the large number of terms and the iterative procedure notwistanding, and even with the complications introduced in later versions (*Milani and Knežević* 1992, 1994), the computational complexity of the fully analytical computation of proper elements is not important (with respect to the computing power available today). To recompute from scratch the proper elements for more than 67,000 asteroids requires currently less than 2 CPU hours on a standard workstation. Thus the only limit to the size of proper elements catalogs, when computed analytically, is the size of the catalogs of osculating elements. Taking into account that asteroids observed only at a single opposition generally have osculating elements affected by uncertainties larger than the inaccuracies in the computation of proper elements, the *AstDys* information system (`http://hamilton.dm.unipi.it/astdys/`) provides analytical proper elements for all numbered and multiopposition asteroids. This catalog is updated every month with the new discoveries, the new numberings, and the new recoveries providing multiopposition orbits. Since the asteroid families are statistical entities, the advantage of such a large set of proper elements may largerly compensate for an accuracy which is not as good as that of the synthetic proper elements.

The second main advantage of the analytical proper elements is that the computational algorithm automatically provides information on all the resonances. Indeed, the iterative procedures used in the computation can be divergent where a *small divisor*, resulting from the attempt to integrate one of the trigonometric terms with very slowly varying arguments, occurs. On the other hand this is, by definition, a resonance. Difficulties in the computation of mean elements (from the osculating ones) are due to mean motion resonances, involving the anomalies of the asteroid and some planet. Difficulties in the convergence of the iterations for the computation of proper elements from mean elements

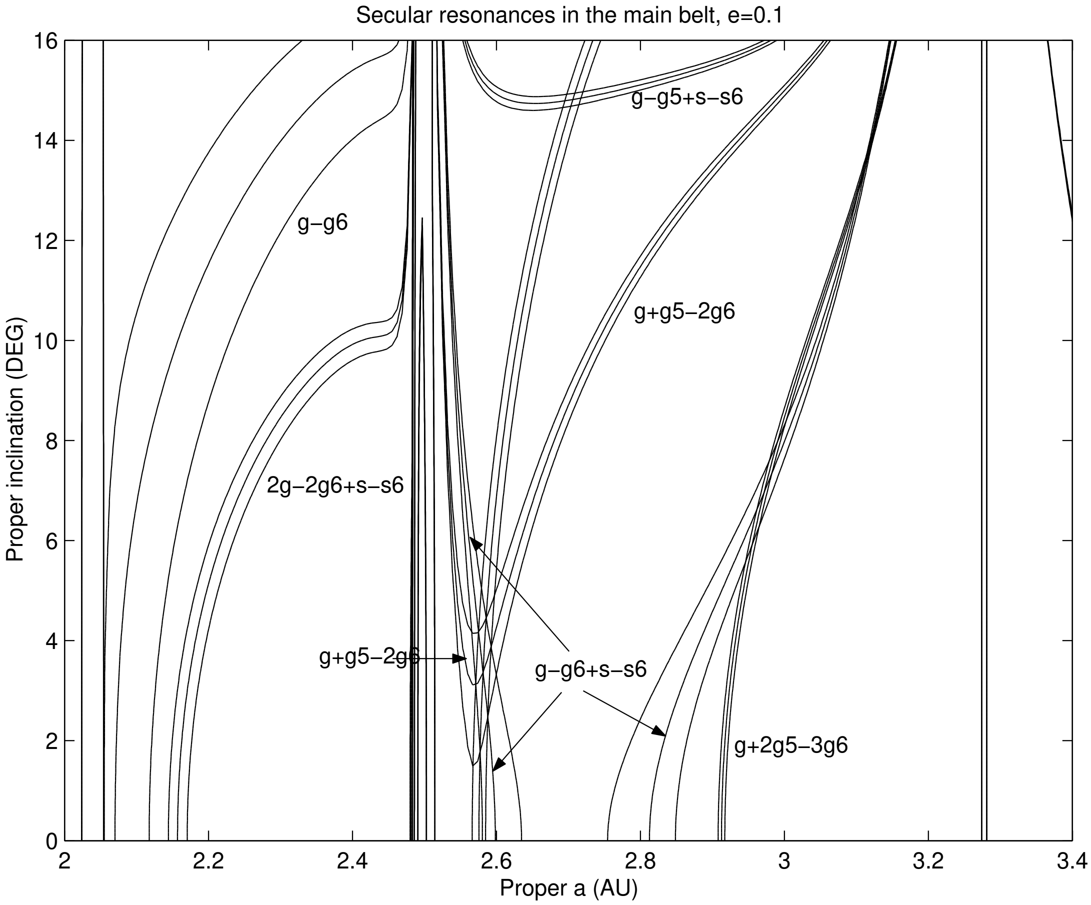

Fig. 2: Map of a sample of secular resonances in the asteroid main belt, involving up to six fundamental frequencies. The locations of the resonances are computed for $e_p = 0.1$, and are marked with three lines corresponding to the zero value of the corresponding divisor, and to ± 0.5 arcsec/yr (with the exception of $g - g_6$ for which we plot wider boundaries corresponding to ± 2 arcsec/yr). g, s are the frequencies of the asteroid perihelion and node, while g_5, g_6, s_6 stand for the corresponding values of Jupiter and Saturn.

indicate secular resonances, that is very small frequencies resulting from combinations of the frequencies g, s of the perihelia and nodes with the corresponding frequencies for the perturbing planets. Thus the analytical proper elements come with quality codes indicating not only lower accuracy due to failure of convergence, but also the resonance responsible for the problem. By the same theories it is possible to construct maps of secular resonances (*Knežević et al.* 1991, *Milani and Knežević* 1994), including combinations of up to four secular frequencies (and in some special cases also 6 and 8 frequencies). In Figure 2 we give an example of such a map with the most important two and four frequencies secular resonances occuring in the main asteroid belt.

2.2 Semianalytical theory

The semianalytic calculation of proper elements was initiated by *Williams* (1969) and then revisited by *Lemaitre and Morbidelli* (1994). It is a classical perturbation method, where the two averaging processes (the first one on the mean longitudes and the second on the pericenters and nodes) are performed numerically. This avoids the expansion in the eccentricity (e) and inclination (i) of the *asteroid*, and makes the method particularly suitable for about 20% of the asteroids with large values of these elements.

The latest version of the theory is written in a Hamiltonian formalism and computed up to the second order in the perturbing masses (presently only Jupiter and Saturn) and up to the first degree in the eccentricities (e') and inclinations (i') *of the perturbers.*

The elimination of the short periodic terms is performed numerically by the calculation of the double integrals (over the two mean longitudes). After this averaging, the semi-major axis is constant and represents the first proper element. To compute the averaged Hamiltonian a Fourier series of the slow angles is used, with the coefficients evaluated on a three dimensional grid (in a, e and i) and stored; a triple linear interpolation is used each time the Hamiltonian and its derivatives have to be evaluated.

The averaged hamiltonian is split in two parts, based upon the smallness of the parameters e' and i'; the integrable problem corresponds to circular and planar motion of the perturbing planets, while the perturbation part gathers all the first order contributions in e' and i'. The dynamics of the integrable problem has been analyzed already by *Kozai*, (1962), and it reveals different behaviors for low and high inclinations; in the latter case, a critical curve separates the phase space into two regions, corresponding to librations (about 90° or 270°) and to circulations of the argument of the pericenter ω of the asteroid.

The removal of the long periodic terms is done by using the action angle variables and is based on the semianalytic method developed by *Henrard* (1990). The resulting hamiltonian K, after the second averaging process, depends only on two proper actions, J and Z, which are both constant. The result is a proper orbit of area proportional to

J, located in a plane identified by the value of Z. Each point on the proper orbit is characterized by a value of the phase ψ, which can be considered as a proper argument of perihelion. From the values of ψ, J and Z, the corresponding values of e, i and ω can be calculated. Any point can be chosen as a "representant" of this orbit : for example, $\psi = 0°, 180°$ corresponds to the minimum value of the eccentricity and maximum of the inclination along the proper orbit. On the other hand, $\psi = 90°, 270°$ corresponds to the maximum of the eccentricity and the minimum of the inclination; this option allows to define proper elements even for ω-librators, since the libration center is either 90° or 270°.

The proper orbit is also characterized by the two basic frequencies, g and s, calculated as the partial derivatives of the Hamiltonian K with respect to the actions J and Z. The set a, g and s is also a set of "proper elements" and is independent of the choice of the representative point on the proper orbit.

The existence of two high inclined groups of asteroids was clearly shown by this method; a Pallas family at about 35° of inclination (*Morbidelli and Lemaitre*, 1994), and a Hungaria family at 2 AU (*Lemaitre*, 1994) .

The precision is limited by the first order development in e' and i' and could not be easily improved. However this semianalytic procedure keeps some advantages, like the fact that it allows the calculation not only of the proper frequencies, but also of their derivatives with respect to the action variables; to our knowledge, this last possibility has not been explored in specific applications.

The last calculation of a complete set of proper elements by the semianalytic method was done in 1994, only for the asteroids for which the analytic method proved not to be applicable. The catalogs with semianalytic elements for about 2500 asteroids are available from: `http://www.fundp.ac.be/sciences/math/equadif.html`.

2.3 Synthetic theory

The latest contribution to the field of asteroid proper element determination is the *synthetic* theory by *Knežević and Milani* (2001). This is, in fact, a set of purely numerical procedures by means of which one can derive classical proper elements $(a_p, e_p, \sin i_p, \varpi_p, \Omega_p)$ and fundamental frequencies (g, s). The theory employs the approach used by *Carpino et* *al.* (1987) for the major planets, and consists of the following steps: first, one numerically integrates the asteroid orbits for a "long enough" time span, together with the orbits of perturbing planets included in the model (the indirect effects of the planets not included in the dynamical model are accounted for by applying the so-called *barycentric correction* to the initial conditions); the short periodic perturbations are removed by means of an online filtering of the osculating elements, performed in the course of the integration itself; simultaneously, the maximum Lyapunov Characteristic Exponents are derived from the variational equations to monitor the chaotic behaviors. The forced oscillations are then removed from the output of the integration and the resulting time series are spectrally resolved under the constraints set by the d'Alembert rules to extract principal harmonics (proper values) together with the associated fundamental frequencies and error estimates (standard deviations and maximum excursions).

Being purely numerical, this theory is comparatively simple in principle. It furnishes results of a superior accuracy with respect to the analytical theory, and, what is of utmost importance, it provides a straightforward way to estimate errors of all the proper values and for each asteroid included in the computation. We used a simple running box method, computing proper values over a number of shorther time intervals and then deriving the corresponding rms and maximum deviation values. The length and shift (i.e. number) of boxes has been selected in such a way to best fit the dynamics, thus in a different way for the inner and the outer belt, for shorther and longer integration time spans, etc.). The theory equally well applies to the asteroids of any eccentricity and inclination, with the obvious exception of the planet crossing objects; since it is based on the averaging principle, it does not work for the resonant asteroids either.

As an example, in Figure 3 the number frequency distributions of error estimates are given for the synthetic proper eccentricities and inclinations of numbered asteroids. These results are on average better by more than a factor of 3 than the typical accuracy of the proper elements derived from the analytical theory (*Milani and Knežević*, 1994).

The numerical integration of tens of thousands of orbits is, on the other hand, a time consuming procedure, so that the synthetic theory is ostensibly less efficient in terms of the CPU time and more demanding in terms of the necessary computing power. This in turn leads to other potential drawbacks like getting results supposedly valid over the very long time spans from the integrations over comparatively short intervals, such short integrations might affect determination of maximum LCE's for the objects subject to a slow chaos, etc.

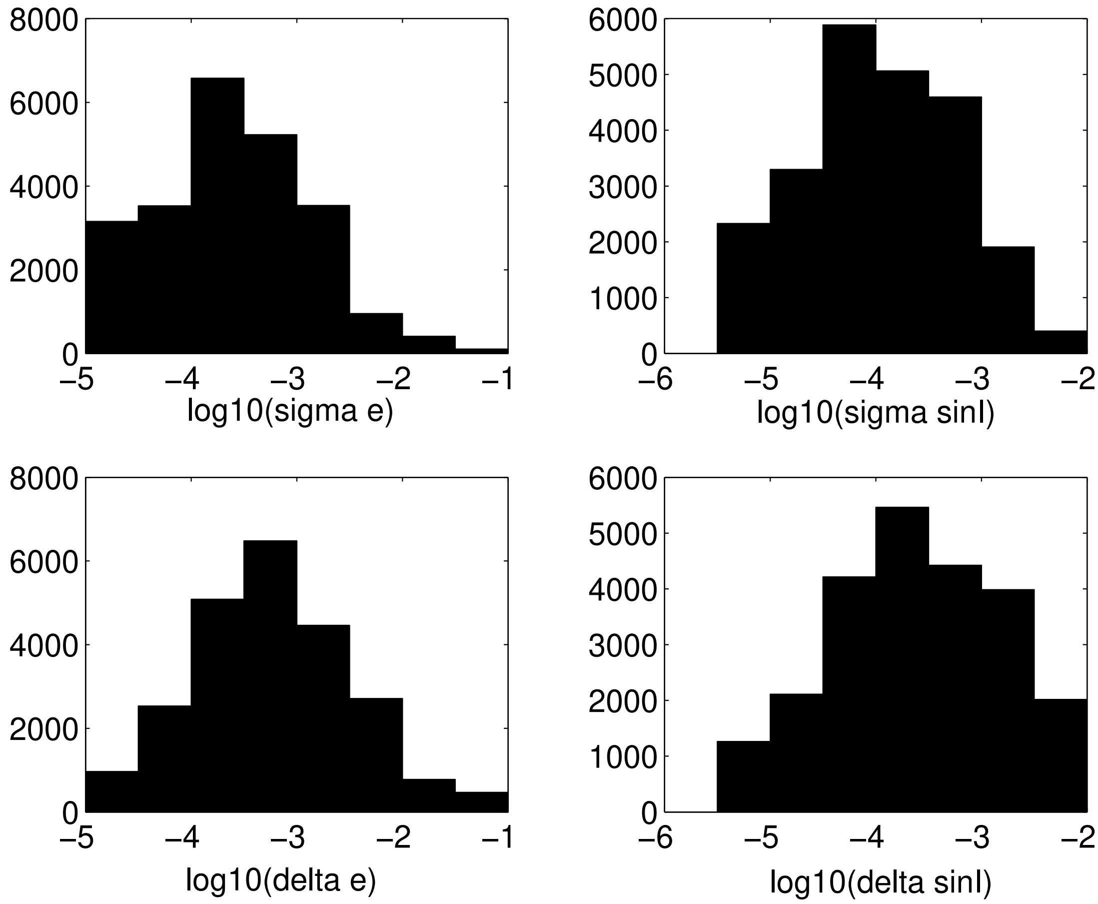

Fig. 3: Number frequency distributions of standard deviations (top) and maximum excursions (bottom) of the proper eccentricity (left) and proper sine of inclination (right). About 80 % of the asteroids have standard deviation of proper eccentricity below 0.001, while for the sine of proper inclination this percentage rises to more than 90 %. Median r.m.s is 0.00024 in proper eccentricity and 0.00011 in sine of proper inclination. Further improvement is still possible, depending on the available computing power.

All these drawbacks are, however, more than amply compensated by the accuracy achieved and by the other above mentioned advantages. The synthetic theory is at the moment certainly the best theory available on the market.

The synthetic proper elements are available from the AstDys service at the address:

http://hamilton.dm.unipi.it/astdys. There are three catalogs ready for the download containing proper elements themselves, their standard deviations and maximum excursions (plus some additional information). These files are updated on a monthly basis for the numbered asteroids between 2.0 and 4.0 AU, with the exception of the planet approaching ones, so that the catalogs as of July 2001, for example, contain proper elements for 24,910 asteroids.

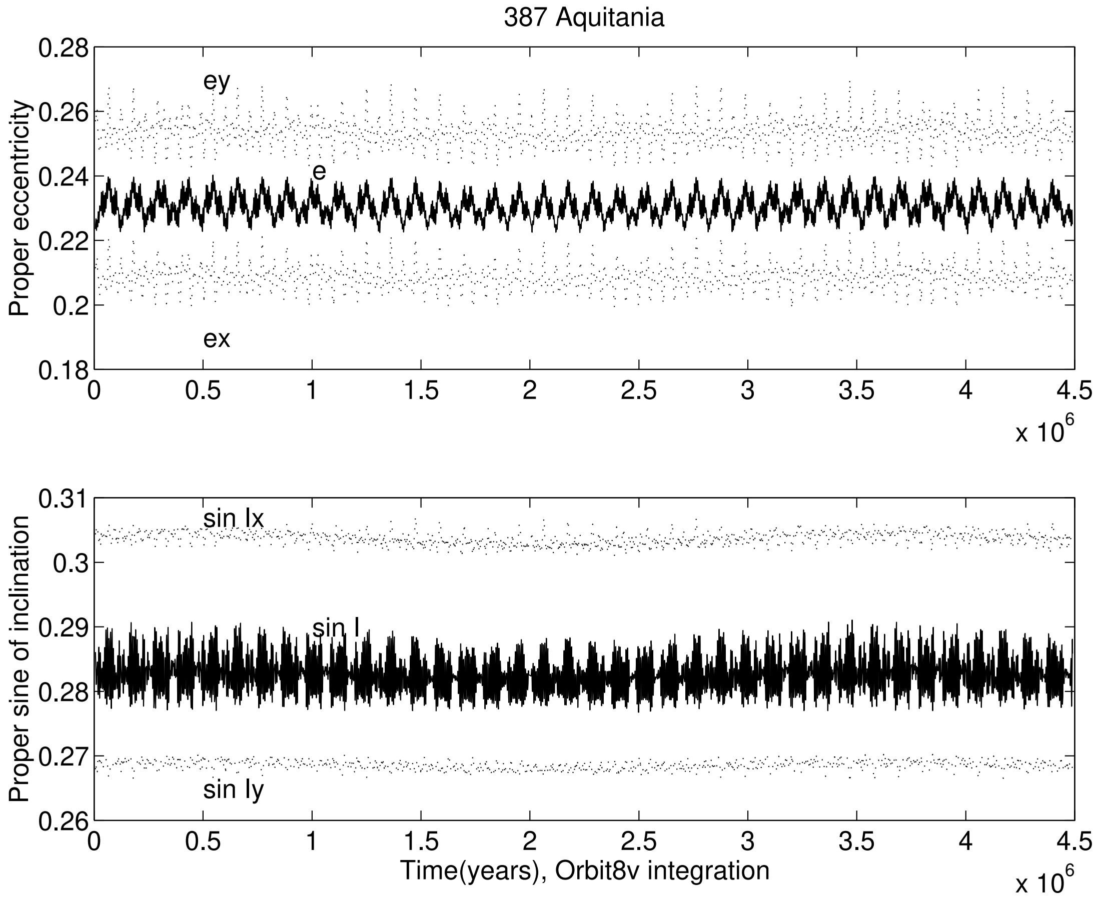

Fig. 4: Comparison of the proper eccentricities (top) and proper inclinations (bottom) for asteroid 387 Aquitania. Proper values determined by means of the semianalytic theory are defined as the values occuring for $\omega = 0°(e_x, I_x)$ and for $\omega = 90°(e_y, I_y)$, thus in the former case corresponding to the minimum of eccentricity and maximum of inclination over a cycle of the argument of perihelion, and to the maximum of e and minimum of I in the latter case. Proper values obtained by means of the analytic theory are in the middle $(e, \sin I)$. For this particular asteroid, having comparatively high orbital inclination, the unremoved instabilities of proper eccentricity as obtained from the two theories are about the same, while for the proper inclination the semianalytical theory already gives better results.

2.4 Comparison of theories in terms of accuracy and efficiency

The only direct comparison between different theories to compute asteroid proper elements has been made by *Knežević et al.* (1995). They integrated the orbits of 5 selected asteroids for 4.5 Myr, simultaneously computing the proper elements and their instabilities by means of the analytical (*Milani and Knežević*, 1994) and semianalytical (*Lemaitre and Morbidelli*, 1994) theories. Comparing the outcomes (see Figure 4 for an example) they concluded that analytic proper elements are more accurate for the asteroids up to about 15° of inclination, while the semianalytic ones are better above about 17°. In the intermediate transition region both methods have roughly the same stability.

There is no direct comparison on a case-by-case basis for the synthetic theory with respect to any of the other theories. Thus, at the moment we can resort only to the general estimate mentioned above of an improvement by a factor of more than 3 on the average, of the synthetic proper elements with respect to the results coming from the analytical theory (Section 2.3).

As for the efficiency in the computation, the analytical theory is by far the fastest, the synthetic theory is the most time consuming, with the semianalytical ones in between. In practice, therefore, when choosing between the theories to be applied under a given circumstances, one always has to find an optimum tradeoff between the conflicting demands of efficiency and accuracy.

3. SPECIAL CASES

3.1 Resonant proper elements

The removal of the short and long periodic terms becomes problematic in the neighborhood of a resonance – a mean motion resonance in the first average procedure, a secular resonance in the second one. If one of the frequencies is close to zero, then the usual averaging methods have to be adapted to this particular situation. The non-resonant harmonics can be eliminated but the critical angle is still present in the averaged problem.

Of course the first reaction is to treat the problem purely numerically, integrating the orbits over millions of years and using a frequency analysis to calculate the proper frequencies. This method seems very suitable, because the real cases of resonant asteroids are not so numerous and the time required by the integration is still reasonable. The Trojans, well known 1:1 jovian resonant asteroids, or the Hilda group were integrated following this idea (Section 3.2).

However the simulations concerning fictitious fragments of asteroids, in order to investigate statistically the efficiency of a particular resonance in delivering meteorites to the Earth for example, require the calculation of thousands of orbits. This encourages the development of specific resonant semi-numerical methods, allowing a picture of the phase space to be drawn and the computation of "resonant proper elements" in a much faster way.

Let us emphasize that these resonant proper elements are not the same as in the non-resonant cases, they can be defined as any identifier of the trajectories obtained in the model after the resonant averaging process. Their definition is arbitrary (for example, the values of the eccentricity or of the inclination when the critical angle equals some specific value, e.g. zero) and cannot be directly connected to the classical non-resonant proper eccentricity and inclination. Thus it is not allowed to mix resonant and non-resonant proper elements for purposes such as family classification.

In this approach, let us mention the calculation of secular resonant proper elements by *Morbidelli* (1993) who applies the same semianalytic method as *Lemaitre and Morbidelli* (1994) but adapted for the resonant case. After the elimination of the short periodic terms, instead of eliminating all secular perturbation terms in the second step, a critical angle ($g - g_6$, for example) is still present in the integrable part of the Hamiltonian. Typical cases such as 582 Olympia, 945 Barcelona or 739 Mandeville have been treated by this method and show a good qualitative agreement with the numerical integrations. However no catalogs or listings of secular resonant real asteroids with corresponding proper elements or identifiers are available; anyone interested in this calculation can contact the author of the method A. Morbidelli (`morby@obs-nice.fr`).

3.2 Trojans, Hildas

The Trojan asteroids are a special case, because they are locked in the 1:1 resonance with Jupiter. Because this is such a strong resonance, librations with very large amplitude are possible: the critical argument $\lambda - \lambda'$ can have small librations around $60°$ and also oscillate between $\simeq 30°$ and $\simeq 100°$. Moreover, the inclination can be very large, up to $33°$. As a result, the development of an analytical theory is very difficult, and indeed it has been achieved only within a simplified 3-body model (*Érdi*, 1988). A semianalytical theory is conceivable, but also difficult to compute within a realistic model.

The synthetic approach has been pioneered by Bien and Schubart (see *Bien and Schubart*, 1987; *Schubart and Bien*, 1997, and the references therein). They use as proper elements a proper eccentricity and a proper inclination defined essentially in the same way as in the synthetic theories for main belt asteroids and an amplitude of libration of $\lambda - \lambda'$. *Milani* (1993, 1994) used a synthetic approach adapted to *Érdi's* (1988) theory, and computed proper elements for 188 numbered, multiopposition and well observed single opposition Trojans. Although the results were suggestive of the possible existence of some Trojan families, the number of asteroids in the catalog was too small for a firm conclusion. *Milani* (1994) also mapped the secular resonances involving the node by fitting the results of the synthetic computations of the frequencies, especially s, with a polynomial in the action variables.

More recently there has been interesting semianalytic work by *Beaugé and Roig* (2001), who produced a catalog (available from the authors upon request) of proper elements for 533 Trojans, proposing also a number of potential dynamical families. These elements are, however, of lower accuracy than those produced by means of a synthetic approach. As of July 2001 there are 679 Trojans with good quality orbits (either numbered or multiopposition), thus the computation of a new catalog of accurate proper elements should allow statistically significant conclusions about the existence of Trojan families.

As regards Hildas, even if important results on their dynamics have been obtained in recent years (e.g. *Ferraz-Mello et al.*, 1998), very little work has been done about their proper elements. The computation of "three characteristic parameters" of orbits of Hilda–type asteroids initiated by *Schubart* (1982), has been later extended to newly discovered objects by the same author (*Schubart*, 1991). Even at present the only source of the data on this group of bodies can be found on the page maintained also by J. Schubart: `http://www.rzuser.uni-heidelberg.de/~s24/hilda.htm`.

4. CONCLUSIONS AND FUTURE WORK

The activity in the field of the determination of asteroid proper elements has been very intense in the previous decade, and one can safely claim that more work has been done and more improvement achieved in this period than in all the preceeding time.

The proper elements are now computable for essentially all asteroids for which there are accurate osculating orbits; thus we currently have proper elements determined for about 70,000 objects. The three principal theories intended for production of large catalogs of proper elements for the main belt objects, plus a number of specially adapted ones for specific groups of objects have been developed and successfully applied in this period.

The accuracy, long-term stability and reliability of the computed proper elements have all remarkably improved (by orders of magnitude). This enabled a lot of important problems to be tackled and provided explanations for many interesting properties of individual asteroids and of their families, which, in turn, decisively contributed to a better understanding of the dynamical and collisional evolution of the asteroid belt as a whole.

Apart from providing a realiable basis for the identification of families, the studies related to the determination of proper elements led to the development of complete theories of asteroid motion, revealing at the same time many novel dynamical features and concepts (non-linear secular resonances, stable chaos, dynamical mass-loss, etc.)

All the main theories to compute asteroid proper elements have now reached a high level of "maturity", being developed almost to their corresponding theoretical limits (*Milani and Knežević*, 1994; see also Section 2.2), while being, on the other hand, amply and thoroughly tested through years of application. Therefore, it is plausible to conjecture that in the near future we can expect more improvement in terms of their practical application, than in the theories themselves.

There are two principal directions in which the development can reasonably take place: improvement of the accuracy and reliability of determination of proper elements (e.g. by extending the time span over which the averaging is done in the synthetic theory), and computation of proper elements for an ever increasing number of asteroids with different dynamical properties (so as to enlarge and complete the database); in this latter class one can include computation of proper elements for objects such as Centaurs and TNO's for which this has not yet been done.

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